Grade 7/8 Math Circles

March 27/28/29/30, 2023

Vectors - Solutions

Exercise Solutions

Exercise 1

Categorize the following as scalar or vector quantities.

a) 2 cups of flour being added to a cookie recipe

b) A thermometer reading 20° Celsius

c) A runner running clockwise around a track

d) A backyard with an area of 50 m^2

e) A car driving 10 km west

f) A cyclist traveling at 30 km/h

g) A cyclist traveling at 30 km/h north

Exercise 1 Solution

a) Scalar

b) Scalar

c) Vector

d) Scalar

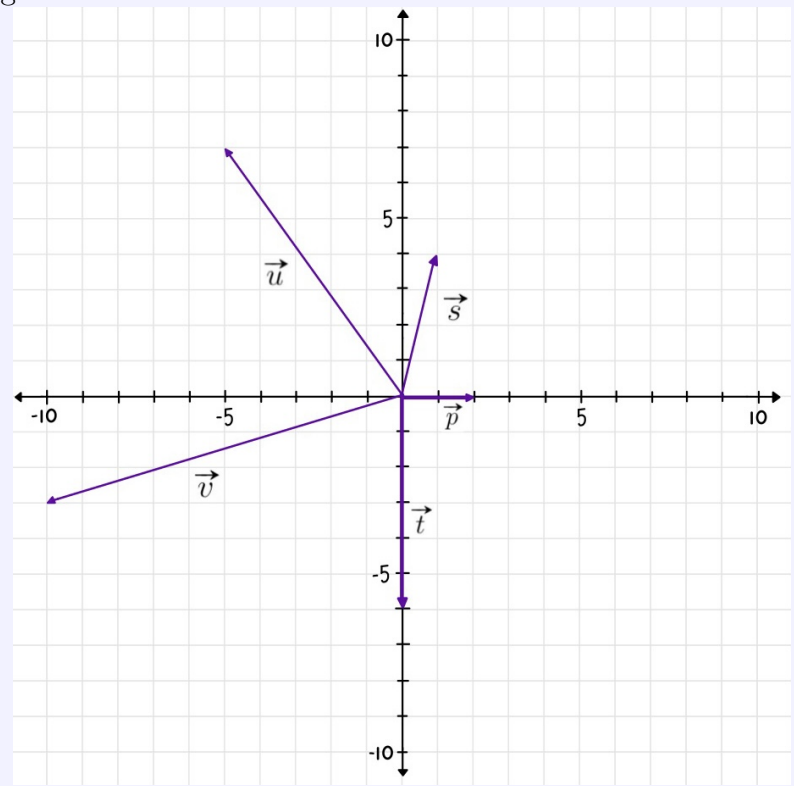
e) Vector

f) Scalar

g) Vector

Exercise 2

Write each of the given vectors in vector notation.



Exercise 2 Solution

- $\vec{v} = \begin{bmatrix} -10 \\ -3 \end{bmatrix}$
- $\vec{u} = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$
- $\vec{s} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$
- $\vec{p} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$
- $\vec{t} = \begin{bmatrix} 0 \\ -6 \end{bmatrix}$

Exercise 3

Find the magnitude of each vector.

a) $\vec{a} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$

b) $\vec{b} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

c) $\vec{c} = \begin{bmatrix} 5 \\ -6 \end{bmatrix}$

d) $\vec{d} = \begin{bmatrix} -10 \\ -1 \end{bmatrix}$

Exercise 3 Solution

a) $\|\vec{a}\| = \sqrt{8^2 + 2^2} = \sqrt{68} \approx 8.25$

b) $\|\vec{b}\| = \sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5$

c) $\|\vec{c}\| = \sqrt{5^2 + (-6)^2} = \sqrt{61} \approx 7.81$

d) $\|\vec{d}\| = \sqrt{(-10)^2 + (-1)^2} = \sqrt{101} \approx 10.05$

Exercise 4

Calculate the following vector addition opperations.

a) $\begin{bmatrix} -4 \\ -2 \end{bmatrix} + \begin{bmatrix} 11 \\ -18 \end{bmatrix}$

b) $\begin{bmatrix} 1 \\ 16 \end{bmatrix} + \begin{bmatrix} -5 \\ 10 \end{bmatrix}$

c) $\begin{bmatrix} 7 \\ 2 \end{bmatrix} + \begin{bmatrix} 12 \\ 15 \end{bmatrix}$

d) $\begin{bmatrix} 6 \\ -8 \end{bmatrix} + \begin{bmatrix} 14 \\ 28 \end{bmatrix}$

Exercise 4 Solution

a) $\begin{bmatrix}-4\\-2\end{bmatrix} + \begin{bmatrix}11\\-18\end{bmatrix} = \begin{bmatrix}(-4)+11\\(-2)+(-18)\end{bmatrix} = \begin{bmatrix}7\\-20\end{bmatrix}$

b) $\begin{bmatrix}1\\16\end{bmatrix} + \begin{bmatrix}-5\\10\end{bmatrix} = \begin{bmatrix}1+(-5)\\16+10\end{bmatrix} = \begin{bmatrix}-4\\26\end{bmatrix}$

c) $\begin{bmatrix}7\\2\end{bmatrix} + \begin{bmatrix}12\\15\end{bmatrix} = \begin{bmatrix}7+12\\2+15\end{bmatrix} = \begin{bmatrix}19\\17\end{bmatrix}$

d) $\begin{bmatrix}6\\-8\end{bmatrix} + \begin{bmatrix}14\\28\end{bmatrix} = \begin{bmatrix}6+14\\(-8)+28\end{bmatrix} = \begin{bmatrix}20\\20\end{bmatrix}$

Exercise 5

Calculate the following vector subtraction operations.

a) $\begin{bmatrix}5\\6\end{bmatrix} - \begin{bmatrix}10\\15\end{bmatrix}$

b) $\begin{bmatrix}-2\\11\end{bmatrix} - \begin{bmatrix}-3\\4\end{bmatrix}$

c) $\begin{bmatrix}-20\\-4\end{bmatrix} - \begin{bmatrix}-18\\-4\end{bmatrix}$

d) $\begin{bmatrix}12\\8\end{bmatrix} - \begin{bmatrix}6\\-1\end{bmatrix}$

Exercise 5 Solution

a) $\begin{bmatrix}5\\6\end{bmatrix} - \begin{bmatrix}10\\15\end{bmatrix} = \begin{bmatrix}5-10\\6-15\end{bmatrix} = \begin{bmatrix}-5\\-9\end{bmatrix}$

b) $\begin{bmatrix}-2\\11\end{bmatrix} - \begin{bmatrix}-3\\4\end{bmatrix} = \begin{bmatrix}(-2)-(-3)\\11-4\end{bmatrix} = \begin{bmatrix}1\\7\end{bmatrix}$

c) $\begin{bmatrix} -20 \\ -4 \end{bmatrix} - \begin{bmatrix} -18 \\ -4 \end{bmatrix} = \begin{bmatrix} (-20) - (-18) \\ (-4) - (-4) \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

d) $\begin{bmatrix} 12 \\ 8 \end{bmatrix} - \begin{bmatrix} 6 \\ -1 \end{bmatrix} = \begin{bmatrix} 12 - 6 \\ 8 - (-1) \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$

Exercise 6

Calculate the following scalar multiplication opperations.

a) $4 \begin{bmatrix} -4 \\ 13 \end{bmatrix}$

b) $\frac{1}{2} \begin{bmatrix} 17 \\ 4 \end{bmatrix}$

c) $-3 \begin{bmatrix} -6 \\ 9 \end{bmatrix}$

d) $-\frac{1}{3} \begin{bmatrix} 15 \\ 0 \end{bmatrix}$

Exercise 6 Solution

a) $4 \begin{bmatrix} -4 \\ 13 \end{bmatrix} = \begin{bmatrix} 4 \times (-4) \\ 4 \times 13 \end{bmatrix} = \begin{bmatrix} -16 \\ 52 \end{bmatrix}$

b) $\frac{1}{2} \begin{bmatrix} 17 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{17}{2} \\ \frac{4}{2} \end{bmatrix} = \begin{bmatrix} 8.5 \\ 2 \end{bmatrix}$

c) $-3 \begin{bmatrix} -6 \\ 9 \end{bmatrix} = \begin{bmatrix} (-3) \times (-6) \\ (-3) \times 9 \end{bmatrix} = \begin{bmatrix} 18 \\ -27 \end{bmatrix}$

d) $-\frac{1}{3} \begin{bmatrix} 15 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{15}{3} \\ -\frac{0}{3} \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \end{bmatrix}$

Exercise 7

Perform the following vector operations to solve for x and y.

a) $\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} 1 \\ 3 \end{bmatrix} - \begin{bmatrix} -5 \\ 8 \end{bmatrix}$

b) $\begin{bmatrix} x \\ y \end{bmatrix} = -3\begin{bmatrix} 2 \\ -5 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 6 \end{bmatrix}$

c) $\begin{bmatrix} -15 \\ -26 \end{bmatrix} = 3\begin{bmatrix} -1 \\ -2 \end{bmatrix} - 4\begin{bmatrix} x \\ y \end{bmatrix}$

d) $\begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix} = -\begin{bmatrix} x \\ y \end{bmatrix} + 4\begin{bmatrix} \frac{1}{3} \\ 2 \end{bmatrix}$

Exercise 7 Solution

a) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2-(-5) \\ 6-8 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$ so $x = 7$ and $y = -2$

b) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6+4 \\ 15+24 \end{bmatrix} = \begin{bmatrix} -2 \\ 39 \end{bmatrix}$ so $x = -2$ and $y = 39$

c) $\begin{bmatrix} -15 \\ -26 \end{bmatrix} = \begin{bmatrix} (-3)-4x \\ (-6)-4y \end{bmatrix}$ so $x = 3$ and $y = 5$

d) $\begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix} = \begin{bmatrix} (-x)+\frac{4}{3} \\ (-y)+8 \end{bmatrix}$ so $x = \frac{5}{6}$ and $y = 6.5$

Exercise 8

Euler is trying to follow directions to get to his friend's house. The directions are as follows:

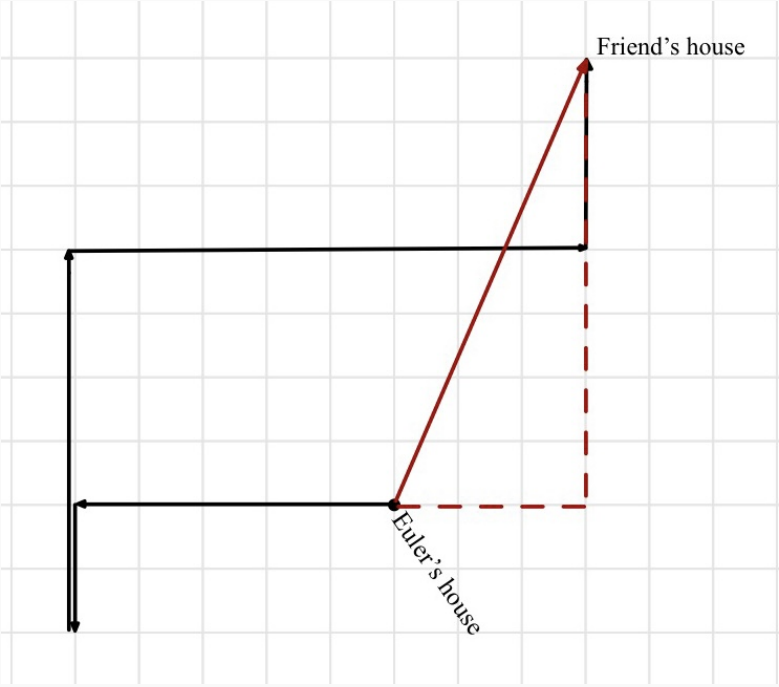
- 5 km west
- 2 km south
- 6 km north
- 8 km east
- 3 km north

(Note that Euler's GPS likes to take him on the most inefficient route).

How far is Euler's house from his friend's house?
What would be a more efficient route for Euler to follow?

Exercise 8 Solution

First, sketch out Euler's path and draw a vector from Euler's house to his friend's house.



We notice that the horizontal distance of this vector is 3 km and the vertical distance is 7 km.

Now, we can calculate the magnitude of the vector:
$\sqrt{3^2 + 7^2} = \sqrt{58} \approx 7.62$ km.
So Euler's friend's house is about 7.62 km from Euler's house. A more efficient route would be to go 3 km east and 7 km north.

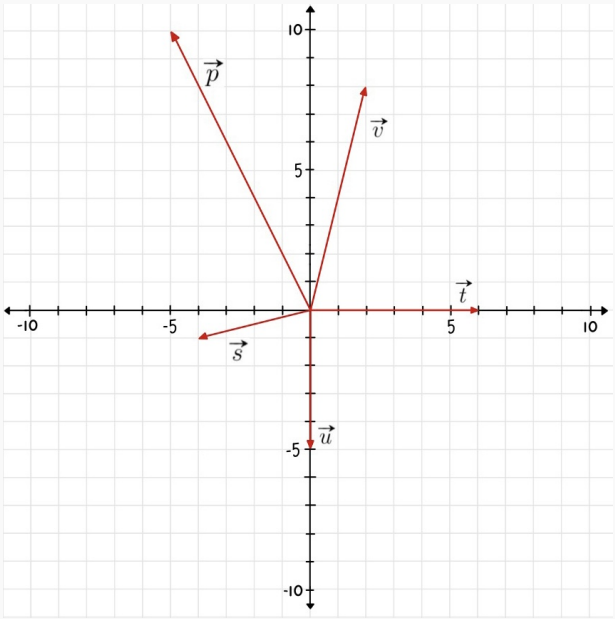
Problem Set Solutions

1. Determine whether the following are scalar or vector quantities.
   a) A car travelling 100 km/h northeast
   b) A 5 kg weight
   c) A 1 km walk to the park
   d) The force of gravity pulling you in a downwards direction
   e) A car accelerating at a rate of 2 m/s^2 south
   f) Finishing a sprint in 15 seconds

   Solution:
   a) Vector
   b) Scalar
   c) Scalar
   d) Vector
   e) Vector
   f) Scalar

2. Draw the following vectors on the Cartesian plane.
   a) $\vec{v} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$
   b) $\vec{u} = \begin{bmatrix} 0 \\ -5 \end{bmatrix}$
   c) $\vec{s} = \begin{bmatrix} -4 \\ -1 \end{bmatrix}$
   d) $\vec{t} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$
   e) $\vec{p} = \begin{bmatrix} -5 \\ 10 \end{bmatrix}$

Solution:



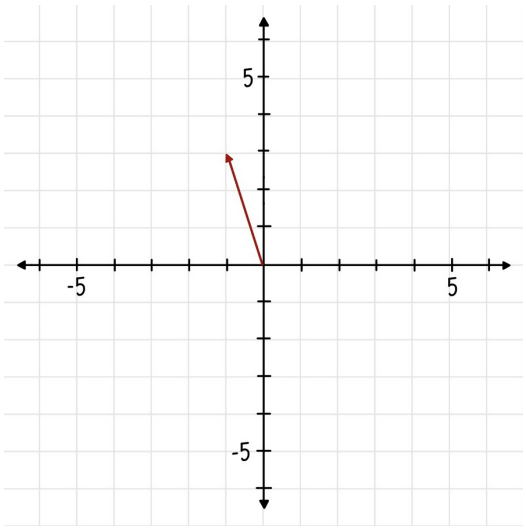
3. Find the magnitude of each vector. Approximate answers to 2 decimal places.

a) $\begin{bmatrix} 1 \\ 7 \end{bmatrix}$

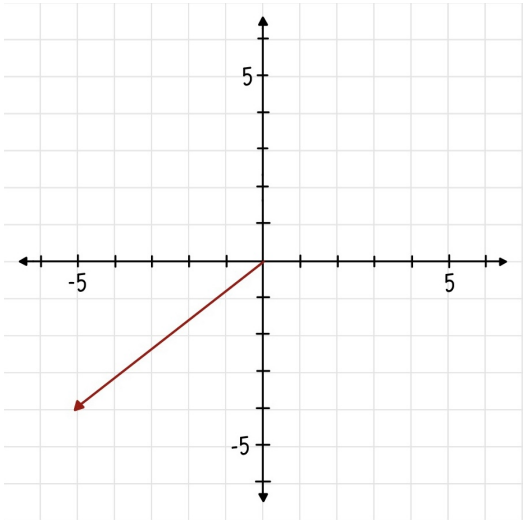
b) $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$

c) $\begin{bmatrix} 5 \\ -12 \end{bmatrix}$

d)



e)



Solution:

a) $\sqrt{1^2+7^2} = \sqrt{15} \approx 3.87$

b) $\sqrt{(-4)^2+2^2} = \sqrt{20} \approx 4.47$

c) $\sqrt{5^2+(-12)^2} = \sqrt{169} = 13$

d) $\sqrt{(-1)^2+3^2} = \sqrt{10} \approx 3.16$

e) $\sqrt{(-5)^2+(-4)^2} = \sqrt{41} \approx 6.40$

4. Find x and y in the following vector equations.

a) $\begin{bmatrix} x \\ y \end{bmatrix} = 12 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} -4 \\ 6 \end{bmatrix}$

b) $\begin{bmatrix} -8 \\ -12 \end{bmatrix} = -4 \begin{bmatrix} 3 \\ 6 \end{bmatrix} + 2 \begin{bmatrix} x \\ y \end{bmatrix}$

c) $\begin{bmatrix} x \\ y \end{bmatrix} = -3 \begin{bmatrix} -8 \\ -7 \end{bmatrix} - 2 \begin{bmatrix} 11 \\ 3 \end{bmatrix}$

d) $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix} - 4 \begin{bmatrix} -4 \\ 2 \end{bmatrix}$

Solution:

a) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12+(-12) \\ 24+18 \end{bmatrix} = \begin{bmatrix} 0 \\ 42 \end{bmatrix}$ so $x = 0$ and $y = 42$

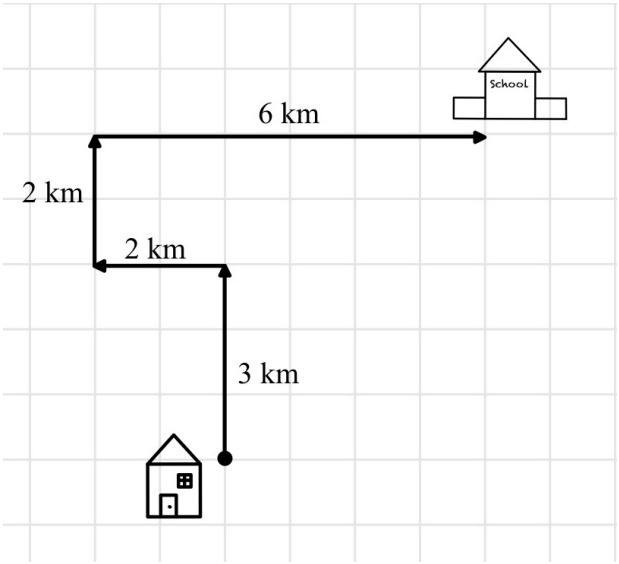
b) $\begin{bmatrix} -8 \\ -12 \end{bmatrix} = \begin{bmatrix} -12+2x \\ -24+2y \end{bmatrix}$ so $x = 2$ and $y = 6$

c) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 24-22 \\ 21-6 \end{bmatrix} = \begin{bmatrix} 2 \\ 15 \end{bmatrix}$ so $x = 2$ and $y = 15$

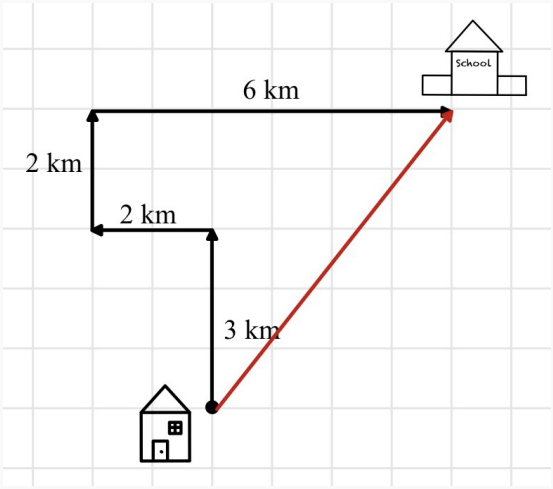
d) $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2x+16 \\ 2y-8 \end{bmatrix}$ so $x = -8$ and $y = 4$

Word Problems

5. Look at the following route that Fibonacci takes to get to school. How far is his house from his school? What direction is his school from his house?



Solution: First, draw a vector from Fibonacci's house to his school.



We notice that the horizontal distance of the vector is 4 km and the vertical distance is 5km. Now, we can calculate the magnitude of the vector:
$\sqrt{4^2+5^2} = \sqrt{41} \approx 6.4$ km
So Fibonacci's school is about 6.4 km from his house and it is in the northeast direction from his house.

6. A motorboat has a speed of 30 km/h in still water. When it moves in the opposite direction of the current, it has a speed of 26 km/h. What is the speed of the current?

Solution: For the boat vector to go from 30 km/h to 26 km/h, it makes sense that 4 km/h is subtracted from the original vector. Thus, the vector of the current is 4 km/h traveling in the opposite direction of the boat.

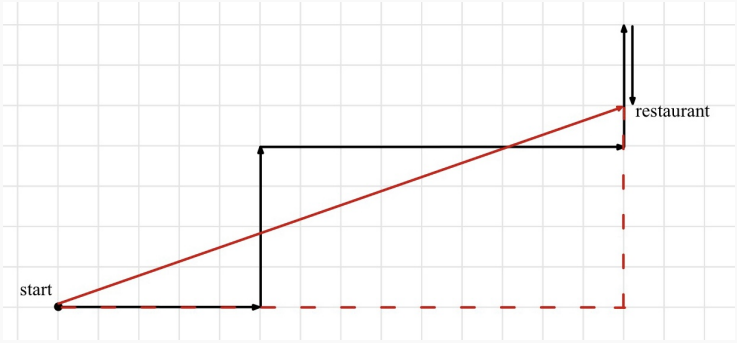
7. Euclid takes the following path to get to his favourite restaurant.

   - 5 km west
   - 4 km north
   - 9 km east
   - 3 km north
   - 2 km south

   Find the distance from Euclid's starting point to the restaurant and state the general direction from Euclid's starting point to the restaurant.

Optional Challenge: Find the angle between the x-axis and the direct path from Euclid's starting point to the restaurant (hint: trigonometry will come in handy here).

Solution: The first step is to sketch out the route and create a vector connecting the starting point to the restaurant.

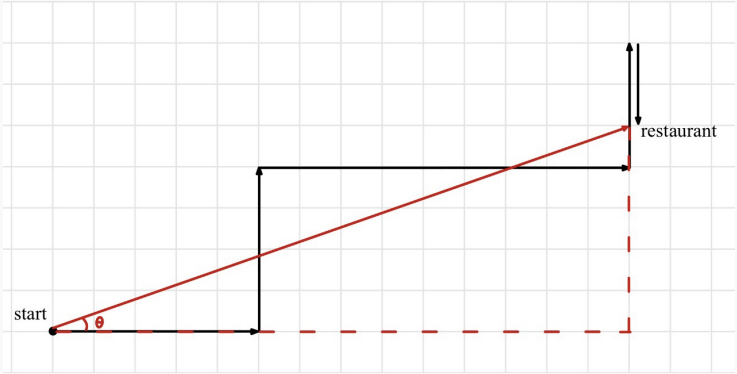


We notice that the horizontal distance of the red vector is 9 km and the vertical distance is 5 km. To find the magnitude of the vector we get:
$\sqrt{9^2 + 5^2} = \sqrt{106} \approx 10.30$ km

So the restaurant is about 10.30 km from Euclid's starting point, pointing in the northeast direction.

Challenge Solution

In the image, we need to find the angle θ.



Using trigonometry, we know that the opposite side is 5 and the adjacent side is 9. So, we can use the $tan(\theta)$ function to find θ.

$$tan(\theta) = \frac{5}{9}$$
$$\theta = tan^{-1}(\frac{5}{9})$$
$$\theta \approx 29.05^\circ$$